SUCCESSIVE-MINIMA-TYPE INEQUALITIES

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Abstract. We show analogues of Minkowski's theorem on successive minima, where the volume is replaced by the lattice point enumerator. We further give analogous results to some recent theorems by Kannan and Lovász on covering minima.

1. Introduction

Throughout this paper E^d denotes the d-dimensional euclidean space and the set of all convex bodies — compact convex sets — in E^d is denoted by $\mathcal{K}^d$. Further $\mathcal{K}_0^d$ denotes the 0-symmetric convex bodies, i.e. $K \in \mathcal{K}^d$ with $K = -K$, and a convex body $K \in \mathcal{K}^d$ is called strictly convex if the segment $\overline{xy}$ intersects the interior of K for all $x, y \in K$, $x \neq y$. The set of lattices $\mathbb{L} \subset E^d$ with lattice determinant $\det(\mathbb{L}) > 0$ is denoted by $\mathcal{L}^d$, and the lattice of all points with integral coordinates in E^d is denoted by $\mathbb{Z}^d$. The k-th coordinate of a point $x \in E^d$ is denoted by x_k, and $\lfloor \alpha \rfloor$ ($\lceil \alpha \rceil$) denotes for $\alpha \in \mathbb{R}$ the largest (smallest) integer $\leq \alpha$ ($\geq \alpha$).

The i-th successive minimum $\lambda_i(K, \mathbb{L})$, $1 \leq i \leq d$, for $K \in \mathcal{K}_0^d$, $\dim(K) = d$, with respect to a lattice $\mathbb{L} \in \mathcal{L}^d$ is defined by

$$\lambda_i(K, \mathbb{L}) = \min\{\lambda \in \mathbb{R} \mid \lambda > 0,\ \dim(\lambda K \cap \mathbb{L}) \geq i\}.$$

Between the volume V and the successive minima Minkowski established for $K \in \mathcal{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathcal{L}^d$ the following relations (cf. [EGH], pp. 28, [GL], pp. 123, [M])

$$(\lambda_1(K, \mathbb{L}))^d V(K) \leq 2^d \det(\mathbb{L}), \tag{1.1}$$

$$\lambda_1(K, \mathbb{L}) \cdot \ldots \cdot \lambda_d(K, \mathbb{L}) V(K) \leq 2^d \det(\mathbb{L}), \tag{1.2}$$

$$\lambda_1(K, \mathbb{L}) \cdot \ldots \cdot \lambda_d(K, \mathbb{L}) V(K) \geq \frac{2^d}{d!} \det(\mathbb{L}). \tag{1.3}$$

All these inequalities are tight. The theorem on successive minima of Minkowski (1.2) is a deep result in geometry of numbers with many applications and is an improvement of (1.1) since $\lambda_1(K, \mathbb{L}) \leq \cdots \leq \lambda_d(K, \mathbb{L})$.

There are several analogues of these results, e.g. by Mahler, Weyl, Hlawka (cf. [EGH], [GL], [H]). In the main part of our paper we give some analogues of (1.1), (1.2) and (1.3) where V is replaced by the lattice point enumerator

$$G(K, \mathbb{L}) = \mathrm{card}(K \cap \mathbb{L}).$$

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The results yield in particular a generalization of the following inequalities by MINKOWSKI ([M], pp. 79) which are closely related to (1.1). For $K \in \mathcal{K}_0^d$, $\mathbb{L} \in \mathcal{L}^d$, $\dim(K) = d$, with $\lambda_1(K, \mathbb{L}) = 1$ holds

$$G(K, \mathbb{L}) \leq 3^d, \tag{1.4}$$

if, in addition, K is strictly convex then

$$G(K, \mathbb{L}) \leq 2^{d+1} - 1. \tag{1.5}$$

The covering minima introduced by KANNAN&LOVÁSZ [KL] form another sequence of numbers associated with a convex body and a lattice. For $K \in \mathcal{K}^d$, $\dim(K) = d$, and $\mathbb{L}\in\mathcal{L}^d$ the i-th covering minimum $\mu_i(K, \mathbb{L})$, $1 \leq i \leq d$, is defined by

$$\mu_i(K, \mathbb{L}) = \min\{t \in \mathbb{R} \mid tK + \mathbb{L} \text{ meets every } (d-i)\text{-dimensional affine subspace}\}.$$

E.g. the last covering minimum $\mu_d(K, \mathbb{L})$ is the classical inhomogeneous minimum (cf. [GL], pp. 98) and $(\mu_1(K, \mathbb{L}))^{-1}$ is called the $\mathbb{L}$-width of K.

KANNAN&LOVÁSZ [KL] showed several analogies and relations between the λ_i and the μ_i. In particular they proved that there are constants $\alpha_d, \beta_d > 0$ only depending on d, such that for $K \in \mathcal{K}^d$ and $\mathbb{L} \in \mathcal{L}^d$ holds:

$$G(K, \mathbb{L}) \geq \left(\left\lfloor \frac{\alpha_d}{\mu_1(K, \mathbb{L})} \right\rfloor \right)^d - 1, \tag{1.6}$$

and if $K \in \mathcal{K}_0^d$

$$G(K, \mathbb{L}) \geq \left\lfloor \left(\frac{\beta_d}{\mu_1(K, \mathbb{L})} - d \right)^d \right\rfloor. \tag{1.7}$$

Here we obtain an analogous result where the lattice point enumerator is replaced by the volume.

2. Lattice Points and Successive Minima

In analogy to (1.1) we have

Theorem 2.1. *Let $K \in \mathcal{K}_0^d$, $\dim(K) = d$, and $\mathbb{L} \in \mathcal{L}^d$. Then*

$$G(K, \mathbb{L}) \leq \left(\left\lfloor \frac{2}{\lambda_1(K, \mathbb{L})} + 1 \right\rfloor \right)^d, \tag{2.1}$$

if K, in addition, is strictly convex

$$G(K, \mathbb{L}) \leq 2 \left(\left\lceil \frac{2}{\lambda_1(K, \mathbb{L})} \right\rceil \right)^d - 1. \tag{2.2}$$

None of these inequalities can be improved.

Remark. Obviously inequality (2.2) is an improvement of (2.1) only if $2/\lambda_1(K,\mathbb{L})$ is an integer.

Proof. It suffices to prove the theorem for the standard lattice $\mathbb{Z}^d$, since $G(K,\mathbb{Z}^d) = G(AK, A\mathbb{Z}^d)$ and $\lambda_i(K,\mathbb{Z}^d) = \lambda_i(AK, A\mathbb{Z}^d)$ for every linear map A with $\det(A) \neq 0$.

Let $p = \lfloor 2/\lambda_1(K,\mathbb{Z}^d) + 1 \rfloor$. First suppose that there are two lattice points $g = (g_1, \dots, g_d)^T$, $h = (h_1, \dots, h_d)^T$, $g \neq h$, in K with

$$g_i \equiv h_i \bmod p, \quad i = 1, \dots, d. \tag{2.3}$$

By the convexity of K and from $p > 2/\lambda_1(K,\mathbb{Z}^d)$ follows that the lattice point

$$\left(\frac{g_1 - h_1}{p}, \dots, \frac{g_d - h_d}{p}\right)^T = \frac{1}{2}\left(\frac{2}{p}g\right) + \frac{1}{2}\left(-\frac{2}{p}h\right) \tag{2.4}$$

belongs to $(\mathbb{Z}^d \backslash \{0\}) \cap \operatorname{int}(\lambda_1(K,\mathbb{Z}^d)K)$.

This is a contradiction to the definition of $\lambda_1(K,\mathbb{Z}^d)$ and so there exist no lattice points $g, h \in K$, $g \neq h$, satisfying (2.3). Hence each lattice point $g \in K$ corresponds uniquely to a representation $(\overline{g}_1, \dots, \overline{g}_d)$ where $\overline{g}_i$ denotes the residue class with respect to p of the i-th coordinate of g. There are at most p^d of such representations, so we get (2.1). For the cube $C_q^d = \{x \in E^d \mid |x_i| \le q,\ 1 \le i \le d\}$, $q \in \mathbb{N}$, follows $G(C_q^d, \mathbb{Z}^d) = (2q+1)^d = (\lfloor 2/\lambda_1(C_q^d,\mathbb{Z}^d) + 1 \rfloor)^d$ and this shows that (2.1) cannot be improved.

For the proof of (2.2) let $p = \lceil 2/\lambda_1(K,\mathbb{Z}^d) \rceil$ and $g, h \in K \cap \mathbb{Z}^d$, $g \neq h$, such that (2.3) holds. From $2/p \le \lambda_1(K,\mathbb{Z}^d)$ follows that the lattice point (2.4) lies in the boundary of $\lambda_1(K,\mathbb{Z}^d)K$. By the strict convexity of K this implies $g = -h$. So (as above) each pair $g, -g$ with $g \in K \cap (\mathbb{Z}^d \backslash \{0\})$ corresponds uniquely to a residue class vector $(\overline{g}_1, \dots, \overline{g}_d)^T$ which shows (2.2). To show that (2.2) is tight, we construct a standard example. Let $C^d = \{x \in E^d \mid 0 \le x_i \le 1,\ 1 \le i \le d\}$ and P the 0-symmetric polytope

$$P = \operatorname{conv}\{C^d, -C^d\}.$$

We have $G(P,\mathbb{Z}^d) = 2^{d+1} - 1$ and $\lambda_1(P,\mathbb{Z}^d) = 1$. With elementary considerations the existence of a strictly 0-symmetric convex body K (in fact of infinitely many) follows with $P \subset K$, $G(K) = G(P)$ and $\lambda_i(K,\mathbb{Z}^d) = 1$, $i = 1, \dots, d$. This shows that (2.2) cannot be improved. q.e.d.

Let us remark that for $\lambda_1(K,\mathbb{Z}^d) = 1$ the inequalities (2.1) and (2.2) become Minkowski's inequalities (1.4) and (1.5). In the case $d = 2$ we can improve (2.1) and (2.2) in the following way

Theorem 2.2. *Let* $K \in \mathcal{K}_0^2$, $\dim(K) = 2$, *and* $\mathbb{L} \in \mathcal{L}^2$. *Then*

$$G(K,\mathbb{L}) \le \left\lfloor \frac{2}{\lambda_1(K,\mathbb{L})} + 1 \right\rfloor \cdot \left\lfloor \frac{2}{\lambda_2(K,\mathbb{L})} + 1 \right\rfloor, \tag{2.5}$$

if K*, in addition, is strictly convex*

$$G(K,\mathbb{L}) \le 2\left\lceil \frac{2}{\lambda_1(K,\mathbb{L})} \right\rceil \cdot \left\lceil \frac{2}{\lambda_2(K,\mathbb{L})} \right\rceil - 1. \tag{2.6}$$

None of these inequalities can be improved.

Remark. Again, in general inequality (2.6) is not an improvement of (2.5).

Proof. It obviously suffices to prove the theorem for the lattice $\mathbb{Z}^2$. Let z^1, z^2 be linearly independent lattice points with $z^i \in \lambda_i(K, \mathbb{Z}^2)K$, $i = 1, 2$, and such that the segment $\overline{z^1 z^2}$ is free of other lattice points. Then the triangle $\mathrm{conv}\{0, z^1, z^2\}$ contains no other lattice points expect $0, z^1, z^2$, and so z^1, z^2 are a basis of $\mathbb{Z}^2$ (cf. [GL], p. 20). Hence we may assume (cf. [GL], p. 22)

$$z^1 = (1,0)^T \quad \text{and} \quad z^2 = (0,1)^T.$$

Now we have for each $x = (x_1, x_2)^T \in K$

$$\lambda_2(K, \mathbb{Z}^2)|x_1| \leq 1 \quad \text{or} \quad \lambda_2(K, \mathbb{Z}^2)|x_2| \leq 1; \tag{2.7}$$

otherwise the lattice point $(x_1/|x_1|, x_2/|x_2|)^T$ would belong to the interior of $\mathrm{conv}(\{\pm z^1, \pm z^2, \lambda_2(K, \mathbb{Z}^2)x\}) \subset \lambda_2(K, \mathbb{Z}^2)K$ which contradicts the definition of the second successive minimum.

Now let $p_i = \lfloor 2/\lambda_i(K, \mathbb{Z}^2) + 1 \rfloor$, $i = 1, 2$, and let $f : E^2 \to E^2$ the linear map with

$$f\left((x_1, x_2)^T\right) = \left(\frac{2}{p_1}x_1, \frac{2}{p_2}x_2\right)^T.$$

With $2/p_i < \lambda_i(K, \mathbb{Z}^2)$ we get from (2.7)

$$f(K) \cap \mathbb{Z}^2 = \{0\}. \tag{2.8}$$

Let $g = (g_1, g_2)^T$, $h = (h_1, h_2)^T$, $g \neq h$, two lattice points of K with

$$g_i \equiv h_i \bmod p_i, \quad i = 1, 2. \tag{2.9}$$

By the convexity of $f(K)$ it follows, that the lattice point

$$\left(\frac{g_1 - h_1}{p_1}, \frac{g_2 - h_2}{p_2}\right)^T = \frac{1}{2}f(g) + \frac{1}{2}f(-h) \tag{2.10}$$

belongs to $f(K) \cap (\mathbb{Z}^2 \backslash \{0\})$ which contradicts (2.8).

Hence there are no two lattice points of K with property (2.9) and so each lattice point $g \in K$ corresponds uniquely to a representation $(\overline{g}_1, \overline{g}_2)$ where $\overline{g}_i$ denotes the residue class with respect to p_i of the i-th coordinate of g. There are at most $p_1 p_2$ of such representations, so we get (2.5).

For the proof of (2.6) let $p_i = \lceil 2/\lambda_i(K, \mathbb{Z}^2) \rceil$, $i = 1, 2$, g, h and $f(K)$ be as above. From $2/p_i \leq \lambda_i(K, \mathbb{Z}^2)$ and (2.7) follows $\mathrm{int}(f(K)) \cap \mathbb{Z}^2 = \{0\}$. Hence the lattice point (2.10) belongs to the boundary of K. With the strict convexity of $f(K)$ this implies $g = -h$ and as in the proof of (2.2) we get (2.6).

The examples in the proof of Theorem 2.1. shows that both inequalities are tight.

q.e.d.

On account of Minkowski's theorem on successive minima (1.2) we conjecture that Theorem 2.2. can be generalized to

Conjecture 2.1. *Let* $K \in \mathcal{K}_0^d$, $\dim(K) = d$, *and* $\mathbb{L} \in \mathcal{L}^d$. *Then*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K, \mathbb{L})} + 1 \right\rfloor ,$$

if K*, in addition, is strictly convex*

$$G(K, \mathbb{L}) \leq 2 \left(\prod_{i=1}^{d} \left\lceil \frac{2}{\lambda_i(K, \mathbb{L})} \right\rceil \right) - 1$$

The following proposition shows that inequalities of this type exist

Proposition 2.1. *Let* $K \in \mathcal{K}_0^d$, $\dim(K)= d$, *and* $\mathbb{L} \in \mathcal{L}^d$. *Then*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left(\frac{2i}{\lambda_i(K, \mathbb{L})} + 1 \right) .$$

Proof. Let $\lambda_1(K, \mathbb{L}), \dots, \lambda_j(K, \mathbb{L}) \leq 1$, $\lambda_{j+1}(K, \mathbb{L}), \dots, \lambda_d(K, \mathbb{L}) > 1$ and let z^1, $\dots, z^j$ be j linearly independent lattice points with $z^i \in (\lambda_i(K, \mathbb{L})K) \cap \mathbb{L}$, $1 \leq i \leq j$. Further let L be the linear subspace spanned by $z^1, \dots, z^j$ and $\overline{K} = K \cap L$, $\overline{\mathbb{L}} = \mathbb{L} \cap L$. We clearly have $\lambda_i(K, \mathbb{L}) = \lambda_i(\overline{K}, \overline{\mathbb{L}})$, $1 \leq i \leq j$, and with Blichfeldt's theorem ([GL], p. 62) and (1.2) for $\overline{K}$, $\overline{\mathbb{L}}$ it follows:

$$G(K, \mathbb{L}) = G(\overline{K}, \overline{\mathbb{L}}) \leq j! \cdot \frac{V_j(\overline{K})}{\det \overline{\mathbb{L}}} + j \leq \prod_{i=1}^{j} \left(\frac{2i}{\lambda_i(K, \mathbb{L})} + 1 \right) ,$$

with equality only for $j = 1$. q.e.d.

On the other hand Conjecture 2.1. is in the sense stronger than Minkowski's second theorem that it is easy to derive the latter from the former:

Proposition 2.2. *If an inequality*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left(\frac{2}{\lambda_i(K, \mathbb{L})} + c_i \right) , \qquad c_i \in \mathbb{R},$$

holds for all $K \in \mathcal{K}_0^d$ *and all* $\mathbb{L} \in \mathcal{L}^d$, *then*

$$\frac{V(K)}{\det(\mathbb{L})} \leq \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \mathbb{L})} .$$

Proof. Let $K \in \mathcal{K}_0^d$. Then we have for $\mu \in \mathbb{R}$, $\mu \neq 0$, $\lambda_i(K, \mu\mathbb{L}) = \mu\lambda_i(K, \mathbb{L})$. Further we have by elementary properties of the Riemann integral

$$\frac{V(K)}{\det(\mathbb{L})} = \lim_{\mu \to 0} \mu^d G(K, \mu\mathbb{L}) \leq \lim_{\mu \to 0} \prod_{i=1}^{d} \mu \left(\frac{2}{\lambda_i(K, \mu\mathbb{L})} + c_i \right) = \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \mathbb{L})} .$$

q.e.d.

The lower bound (1.3) is much easier to prove than Minkowski's theorem (1.2). The same seems to hold for the case of lattice points as we have as satisfactory general lower bound:

Theorem 2.3. *Let* $K \in \mathcal{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathcal{L}^d$, *and* $\lambda_1(K, \mathbb{L}) \leq 1$. *Then*

$$G(K, \mathbb{L}) \geq \frac{2^d}{d!} \left(1 - \frac{\lambda_1(K, \mathbb{L})}{2}\right)^d \prod_{i=1}^{d} \frac{1}{\lambda_i(K, \mathbb{L})}. \tag{2.11}$$

In general the constants cannot be improved.

Proof. Again, it suffices to prove the theorem for the lattice $\mathbb{Z}^d$. For convenience we write $\nu_i = 1/\lambda_i(K, \mathbb{Z}^d)$, $1 \leq i \leq d$. Let $z^1, \ldots, z^d$ be d linearly independent points in $\mathbb{Z}^d$ with $\nu_i z^i \in K$, $1 \leq i \leq d$, and let $Q \subset K$ be the crosspolytope with vertices $\pm\nu_i z^i$, $1 \leq i \leq d$. Denoting by e^i the i-th coordinate unit vector and by P the crosspolytope with vertices $\pm\nu_i e^i$, $1 \leq i \leq d$, we have $G(K, \mathbb{Z}^d) \geq G(Q, \mathbb{Z}^d) \geq G(P, \mathbb{Z}^d)$ and $V(P) = 2^d/d!(\nu_1 \cdot \ldots \cdot \nu_d)$. Hence it is only necessary to prove

$$G(P, \mathbb{Z}^d) \geq \left(1 - \frac{1}{2\nu_1}\right)^d V(P). \tag{2.12}$$

Let $\rho = 1 - \frac{1}{2\nu_1}$ and E_j denote the plane spanned by $e^1, \ldots, e^j$, $1 \leq j \leq d$. The total orthogonal complement of the plane E_j is denoted by $E_j^{\perp}$. We show by induction that for each $z^j \in \rho P \cap \mathbb{Z}^d \cap E_j^{\perp}$

$$G(P \cap (z^j + E_j), \mathbb{Z}^d) \geq V^j(\rho P \cap (z^j + E_j)), \tag{2.13}$$

where V^j denotes the j-dimensional volume. For $j = 1$ we have $V^1(\rho P \cap (z^1 + E_1)) = V^1(P \cap (z^1 + E_1)) - 1$. As for any segment S we have $G(S, \mathbb{Z}^d) \geq V^1(S) - 1$ the assertion is proved. Now let $z^{j+1} \in \rho P \cap \mathbb{Z}^d \cap E_{j+1}^{\perp}$ and let $\eta \in \mathbb{R}$ be the maximal number such that $\eta e^{j+1} + z^{j+1} \in \rho P$. Then we have

$$\begin{aligned} G(P \cap (z^{j+1} + E_{j+1}), \mathbb{Z}^d) &\geq \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} G(P \cap (ie^{j+1} + z^{j+1} + E_j), \mathbb{Z}^d) \\ &\geq \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} V^j \left(\rho P \cap (ie^{j+1} + z^{j+1} + E_j)\right) \\ &= V^j(\rho P \cap (z^{j+1} + E_j)) \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} \left(1 - \frac{|i|}{\eta}\right)^j. \end{aligned}$$

It follows

$$G(P \cap (z^{j+1} + E_{j+1}), \mathbb{Z}^d) \geq 2V^j(\rho P \cap (z^{j+1} + E_j)) \left(\sum_{i=0}^{\lfloor\eta\rfloor} \frac{f(i) + f(i+1)}{2}\right), \tag{2.14}$$

where f is the function given by

$$f(x) = \begin{cases} \left(1 - \frac{x}{\eta}\right)^j, & \text{for } x \in (-\infty, \eta] \\ 0, & \text{for } x \in [\eta, \infty). \end{cases}$$

Now f is convex on $[0,\infty)$ and from this we obtain by elementary properties of the integral

$$\sum_{i=0}^{\lfloor\eta\rfloor} \frac{f(i)+f(i+1)}{2} \geq \int_0^\eta f(x)dx = \frac{\eta}{j+1}.$$

Along with (2.14) it follows

$$\begin{aligned} G(P\cap(z^{j+1}+E_{j+1}),\mathbb{Z}^d) &\geq \frac{2\eta}{j+1}V^j(\rho P\cap(z^{j+1}+E_j)) \\ &= V^{j+1}(\rho P\cap(z^{j+1}+E_{j+1})), \end{aligned}$$

which proves (2.13). In particular for $j=d$ (2.13) is equivalent to (2.12).

It remains to show that (2.11) cannot be improved in general. To this end we consider the regular crosspolytope $P^d=\mathrm{conv}(\{\pm e^1,\dots,\pm e^d\})$. From Ehrhart's theorems (cf. [GL], pp. 135) follows that for $k\in\mathbb{N}$

$$G(kP^d,\mathbb{Z}^d)=\sum_{i=0}^d k^i G_i(P^d),\quad G(kP^d,\mathbb{Z}^d)^\circ=\sum_{i=0}^d k^i(-1)^{d-i}G_i(P^d), \tag{2.15}$$

where $G(P^d,\mathbb{Z}^d)^\circ$ denotes the number of lattice points in the interior of P^d and $G_i(P^d)$ are constants. Especially we have $G_d(P^d)=V(P^d)=2^d/d!$. Next we observe

$$G(kP^d,\mathbb{Z}^d)=G(kP^d,\mathbb{Z}^d)^\circ+G(kP^{d-1},\mathbb{Z}^d)^\circ+G(kP^{d-1},\mathbb{Z}^d)$$

and thus we obtain from (2.15) $G_{d-1}(P^d)=\frac{2^{d-1}}{(d-1)!}$. On the other hand the formula (2.11) yields

$$G(kP^d,\mathbb{Z}^d)\geq\left(1-\frac{1}{2k}\right)^d V(kP^d)=\frac{2^d}{d!}k^d\sum_{i=0}^d\binom{d}{i}\left(\frac{-1}{2k}\right)^i.$$

Comparing the coefficients of k^d and k^{d-1} with the coefficients of $G(kP^d,\mathbb{Z}^d)^\circ$ in (2.15) we see, that we can do no better than in the theorem. q.e.d.

Minkowski's inequalities (1.2) and (1.3) appear to be much more symmetric than Theorem 2.2. and Theorem 2.3. By a slight weakening of Theorem 2.3. we obtain a corollary, which is up to the Gauss-brackets completely symmetric to Theorem 2.2 in the same way as (1.2) to (1.3):

Corollary 2.1. *Let $K\in\mathcal{K}_0^d$, $\dim(K)=d$, $\mathbb{L}\in\mathcal{L}^d$, and $\lambda_d(K,\mathbb{L})\leq 2$. Then*

$$G(K,\mathbb{L})\geq\frac{1}{d!}\prod_{i=1}^d\left(\frac{2}{\lambda_i(K,\mathbb{L})}-1\right).$$

In general the constants cannot be improved.

Proof. On account of $\lambda_1(K,\mathbb{L})\leq\cdots\leq\lambda_d(K,\mathbb{L})$ the assertion follows from (2.11). q.e.d.

In the proof of Theorem 2.3. (2.12) appears to have some interest of its own, as it relates volume, lattice number and successive minima for crosspolytopes. Thus there is the natural question for a formula of this kind, which holds for all 0-symmetric convex bodies, and for a corresponding upper bound. Certainly (2.12) is not true for all $K\in\mathcal{K}_0^d$ as the class of open boxes with edges parallel to the coordinate axes shows. But this class suggests:

Conjecture 2.2. *Let $K \in \mathcal{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathcal{L}^d$, and $\lambda_d(K, \mathbb{L}) \leq 2$. Then*

$$\frac{V(K)}{\det(\mathbb{L})} \prod_{i=1}^{d} \left(1 - \frac{\lambda_i(K, \mathbb{L})}{2}\right) \leq G(K, \mathbb{L}).$$

3. Covering Minima

For the volume of a convex body we have the following lower bound with respect to the covering minima

Theorem 3.1. *Let $K \in \mathcal{K}^d$ and $\mathbb{L} \in \mathcal{L}^d$. Then there is a constant τ_d, only depending on d, with $0 < \tau_d \leq (d!)^{-1}$ and*

$$(\mu_1(K, \mathbb{L}))^d \, V(K) \geq \tau_d \cdot \det(\mathbb{L}). \tag{3.1}$$

Proof. Since $V(K) = V(A^{-1}K) \cdot |\det(A)|$ and $\mu_i(K, \mathbb{L}) = \mu_i(A^{-1}K, A^{-1}\mathbb{L})$ for every linear map A with $\det(A) \neq 0$ it suffices to prove the theorem for the lattice $\mathbb{Z}^d$. For a convex body $K \in \mathcal{K}^d$ Kannan&Lovász [KL] proved

$$\mu_1(K, \mathbb{Z}^d) = (\lambda_1((K-K)^*, \mathbb{Z}^d))^{-1},$$

where $(K-K)^*$ denotes the polar body of the difference body $K-K$ of K. So

$$\mu_1(K, \mathbb{Z}^d)^d V(K) = \left(\lambda_1((K-K)^*, \mathbb{Z}^d)\right)^{-d} V(K). \tag{3.2}$$

From Roger's and Shephard's theorem on the difference body (cf. [GL], p. 32) and Bourgain's and Milman's theorems on the polar body (cf. [EGH], p. 31, [KL]) we have with a constant c_1

$$\binom{2d}{d} V(K) \geq V(K-K) \geq \left(\frac{c_1}{d}\right)^d V((K-K)^*)^{-1} \tag{3.3}$$

From (3.2) and (3.3) we obtain with (1.1) and $\binom{2d}{d}^{-1} \left(\frac{c_1}{d}\right)^d = 2^d \tau_d$

$$\mu_1(K, \mathbb{Z}^d)^d V(K) \geq 2^d \tau_d \left(\lambda_1((K-K)^*, \mathbb{Z}^d)\right)^{-d} V((K-K)^*)^{-1} \geq \tau_d.$$

The regular crosspolytope shows that $\tau_d \leq (d!)^{-1}$. q.e.d.

The constants α_d, β_d in (1.6) , (1.7) and τ_d in (3.1) are not best possible. We conjecture

Conjecture 3.1. *Let $K \in \mathcal{K}^d$ and $\mathbb{L} \in \mathcal{L}^d$. Then*

$$(\mu_1(K, \mathbb{L}))^d \, V(K) \geq \frac{\det(\mathbb{L})}{d!}.$$

From $\mu_1(K, \mathbb{L}) \leq \cdots \leq \mu_d(K, \mathbb{L})$ and (2.6) follows

$$\mu_1(K, \mathbb{L}) \cdot \ldots \cdot \mu_d(K, \mathbb{L}) V(K) \geq \tau_d \cdot \det(\mathbb{L}),$$

i.e. an analogue of Minkowski's theorem on successive minima (1.2), although not with best constant. As a direct consequence of a result by Nosarzewska, Hadwiger and Wills one obtains for the surface area F and the lattice $\mathbb{Z}^d$

Proposition 3.1. *Let* $K \in \mathcal{K}^d$, $\dim(K) = d$, *and* $1 \le i \le d$. *Then*

$$\mu_i(K, \mathbb{Z}^d)V(K) < \frac{1}{2}F(K). \tag{3.4}$$

None of this inequalities can be improved.

Proof. For lattice-point-free $K \in \mathcal{K}^d$ with respect to the standard lattice $\mathbb{Z}^d$ Nosarzewska ($d = 2$), Wills ($d = 3, 4$) and Hadwiger (general d) (cf. [GL], p. 282, [EGH], p. 22) proved $V(K) < \frac{1}{2}F(K)$. From this follows for general $K \in \mathcal{K}^d$

$$\mu_d(K, \mathbb{Z}^d)V(K) < \frac{1}{2}F(K).$$

On account of $\mu_i(K, \mathbb{Z}^d) \le \mu_d(K, \mathbb{Z}^d)$, $1 \le i \le d$, shows this (3.4). Now let $q \in \mathbb{N}$, $q \ge 3$, and

$$Q_q = \left\{ x \in E^d \mid |x_i| \le q,\, 1 \le i \le d-1,\, |x_d| \le \frac{1}{2} - \frac{1}{q} \right\}.$$

Then $\mu_1(Q_q, \mathbb{Z}^d) > 1$, $V(Q_q) < \frac{1}{2}F(Q_q)$ and

$$\lim_{q \to \infty} \mu_1(Q_q, \mathbb{Z}^d)V(Q_q)F(Q_q)^{-1} = \frac{1}{2},$$

hence none of the inequalities can be improved. q.e.d.

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